A STUDY ON SOME NEW RESULTS ARISING FROM (p,q)-CALCULUS

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ABSTRACT. This paper includes some new investigations and results for post quantum calculus, denoted by (p,q)-calculus. A chain rule for (p,q)-derivative is given. Also, a new (p,q)-analogue of the exponential function is introduced and its properties including the addition property for (p,q)-exponential functions are investigated. Several useful results involving (p,q)binomial coefficients and (p,q)-antiderivative are discovered. At the final part of this paper, (p,q)-analogue of some elementary functions including trigonometric functions and hyperbolic functions are considered and some properties and relations among them are analyzed extensively.

Keywords: q-calculus, (p,q)-calculus, exponential functions, trigonometric functions, hyperbolic functions.

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1. INTRODUCTION

As q-calculus including q-numbers with one base q is dealt with by some scientists since XIX century, (p,q)-calculus including (p,q)-number with independent two variables p and q are firstly considered circa the same time (1991) and independently by [7], et al. [5], [28], et al. [4]. [7] introduced the (p,q)-number to generalize or unify several forms of q-oscillator algebras well-known in the physics literature related to the representation theory of single parameter quantum algebras. [5] introduced (p,q)-number so that (p,q)-algebra can be derived from q-calculus by a Bargmann differential realization of the creation and annihilation operator, the Bose representation of those operators was derived, and (p,q)-Harmonic oscillator was constructed. White [28] introduced (p,q) -number in mathematical literature in order to obtain (p,q)-Stirling number which is the generating function of the joint distribution of pairs of statistics. [4] introduced (p,q)-number to investigate Fibonacci oscillators. Thereby, several physical and mathematical problems lead to the necessity of (p,q)-calculus. Based on the aforementioned papers, many mathematicians and physicists have developed the (p,q)-calculus in many different research areas since 1991. For instance, [6] introduced (p,q)-hypergeometric functions and studied the relations among the basic hypergeometric functions, q-hypergeometric functions, and (p,q)-hypergeometric functions in 1994. [17] considered a more general (p,q)-hypergeometric

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series as well as Burban's (p,q)-hypergeometric series in [6] and he derived some related preliminary results in 1997. In 2005, [16] gave a method to embed the q-series into a (p,q)-series, discovered some results corresponding (p,q)-extensions of the known q-identities and they also studied about (p,q)-hypergeometric series, based on the (p,q)-numbers. In 2008, [10] developed the (p,q)-extension of the binomial coefficients and also established some properties parallel to those of the ordinary and q-binomial coefficients. [25] investigated some properties of the (p,q)-derivative and the (p,q)-integration and presented two appropriate polynomial bases for the (p,q)-derivative, and then derived various properties of these bases in the year 2013. As an application, he gave two (p,q)-Taylor formulas for polynomials. Furthermore, the fundamental theorem of (p,q)-calculus and the formula of (p,q)-integration by part were given. In the year 2015, [26] introduced a new generalization of the Gamma and the Beta functions, calling as (p,q)-Gamma and (p,q)-Beta functions and developed some of their properties reduced to the known results as special cases.

Recently, (p,q)-integers have been introduced into classical linear positive operators to construct new approximation processes. A sequence of (p,q)-analogue of Bernstein operators was first introduced by [20]. For further developments, one can also refer to [1], [21], [22], [23].

The (p,q)-analogues of Bernoulli polynomials, Euler polynomials, and Genocchi polynomials were described by et al. [11] in early 2016 and the (p,q)-analogues of known earlier formulae were obtained, and relations between the new and old polynomials were investigated by making use of the fermionic p-adic integral over the p-adic number fields. Also, in early 2016, a new class of Bernoulli, Euler and Genocchi polynomials based on the theory of (p,q)-calculus were considered by et al. [13] some of their properties including addition theorems, difference equations, derivative properties, recurrence relationships were investigated. The (p,q)-extension of main result in [8] was acquired, and further, the (p,q)-analogue of the main results given earlier by [27] was discovered in [13]. et al. [3] researched and gave some connections between the (p,q)-derivative operators and divided differences in 2016. Moreover, in 2016, et al. [2] considered an extension of Haar distribution based on (p,q)-numbers. By means of this distribution, the (p,q)-analogue of Volkenborn integration that is a new generalization of q-Volkenborn integration was derived. Some properties of Volkenborn integration based on (p,q)-numbers were investigated. Finally, (p,q)-Bernoulli numbers and polynomials derived from (p,q)-Volkenborn integral were constructed and also some of their properties were obtained.

We now review briefly some concepts of the (p,q)-calculus.

We begin with the following notations: $\mathbb{N}$ denotes the set of the natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

The twin-basic number or (p,q)-number is defined by, for any number n,

$$[n]_{p,q} := \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$$

which is a natural generalization of the q-number such that

$$[n]_{1,q} := [n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-2} + q^{n-1},$$

Note that $[n]_{p,q} = [n]_{q,p}$.

The (p,q)-derivative of a function f with respect to x is defined by

$$D_{p,q;x}f(x) := D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x} \qquad (x \neq 0), \tag{1}$$

and $(D_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0. As with the q-derivative and the ordinary derivative, the action of applying the (p,q)-derivative of any function is a linear operator, viz., for any constants a and b,

$$D_{p,q}(af(x) + bg(x)) = aD_{p,q}f(x) + bD_{p,q}g(x).$$

The (p,q)-derivatives of product and the quotient of $f(x)$ and $g(x)$ are given by

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \quad (2)$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \quad (3)$$

As well as the formulas (1.4) and (1.5) we may write one more representation in symmetrical form

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{1}{2}\frac{D_{p,q}f(x)(g(px) + g(qx)) - D_{p,q}g(x)(f(px) + f(qx))}{g(px)g(qx).} \quad (4)$$

The formulas in Eqs. (1)-(4) are valid, however one of these forms may be more useful than others under special cases.

The (p,q)-Gauss Binomial formula is defined by

$$(x \oplus a)_{p,q}^n = \begin{cases} (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \end{cases}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k},$$

where the notations $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ and $[n]_{p,q}!$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}, \quad (n \geq k),$$

and

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}, \quad (n \in \mathbb{N}).$$

The (p,q)-exponential functions, $e_{p,q}(x)$ and $E_{p,q}(x)$, are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!},$$

which hold the basic identity

$$e_{p,q}(x)E_{p,q}(-x) = 1. \quad (5)$$

The following (p,q)-derivatives hold true:

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \quad and \quad D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \quad (6)$$

Note that the (p,q)-derivatives of the (p,q)-exponentials are not precisely determined. However those derivatives are in the similar form of the derivatives of classical exponential functions.

A more detailed statement of above, including (p,q)-numbers, is found in [1-7, 10, 11, 13, 16, 17, 20-23, 25].

Taking here $p = 1$, then all notations given in this part are reduce to the notations of the usual q-calculus (for details see [9, 12, 14, 15, 18, 19, 21]).

2. Main Results

The formulas (3) and (4) can be rewritten in explicitly symmetrical form:

$$D_{p,q}\left(f\left(x\right)g\left(x\right)\right) = D_{p,q}f\left(x\right)\left(\frac{g\left(px\right) + g\left(qx\right)}{2}\right) + D_{p,q}g\left(x\right)\left(\frac{f\left(px\right) + f\left(qx\right)}{2}\right). \tag{7}$$

More general form of the multiplication rule of (p,q)-derivative is presented with fixed $0 \le \alpha \le 1$,

$$D_{p,q}\left(f\left(x\right)g\left(x\right)\right) = \left(\alpha f\left(qx\right) + \left(1 - \alpha\right)f\left(px\right)\right)D_{p,q}g\left(x\right) + \left(\alpha g\left(px\right) + \left(1 - \alpha\right)g\left(qx\right)\right)D_{p,q}f\left(x\right).$$

If we choose $\alpha = 1$, $\alpha = 0$ and $\alpha = \frac{1}{2}$, we then get the formulas (3), (4) and (7), respectively.

Let $y = f\left(x\right)$ be an injective and surjective mapping. In this case, we have $x = f^{-1}\left(y\right)$ where f^{-1} is the inverse function of f. Applying (p,q)-derivative to each side of $x = f^{-1}\left(y\right)$ gives

$$\begin{aligned} 1 &= D_{p,q}x = D_{p,q}f^{-1}\left(y\right) \\ &= \frac{f^{-1}\left(y\left(px\right)\right) - f^{-1}\left(y\left(qx\right)\right)}{y\left(px\right) - y\left(qx\right)} \cdot \frac{y\left(px\right) - y\left(qx\right)}{px - qx} \\ &= D_{p,q;x}f^{-1}\left(y\left(x\right)\right) \cdot D_{p,q;x}y\left(x\right), \end{aligned}$$

thus we arrive

$$D_{p,q;x}f^{-1}\left(y\left(x\right)\right) = \frac{1}{D_{p,q;x}y},$$

which is (p,q)-extension of the usual derivative of inverse function f^{-1}.

As it has been given for q-derivative in [18], there doesn't exist a general chain rule for (p,q)-derivatives. That is, if we consider the function $f(u(x))$, where $u = u(x) = \lambda x^{\mu}$ with λ, μ being constants, we have a chain rule as a special case:

$$\begin{aligned} D_{p,q}\left[\,f(u(x))\right] &= D_{p,q}\left[f\left(\lambda x^{\mu}\right)\right] = \frac{f(\lambda x^{\mu}p^{\mu}) - f(\lambda x^{\mu}q^{\mu})}{x(p - q)} \\ &= \frac{f(\lambda x^{\mu}p^{\mu}) - f(\lambda x^{\mu}q^{\mu})}{\lambda x^{\mu}p^{\mu} - \lambda x^{\mu}q^{\mu}} \cdot \frac{\lambda x^{\mu}p^{\mu} - \lambda x^{\mu}q^{\mu}}{x(p - q)} \\ &= \frac{f(up^{\mu}) - f(uq^{\mu})}{up^{\mu} - uq^{\mu}} \cdot \frac{u(px) - u(qx)}{x(p - q)}, \end{aligned}$$

which gives

$$D_{p,q}\;f(u(x)) = \left(D_{p^{\mu},q^{\mu}}f\right)(u(x)).D_{p,q}u(x), \tag{8}$$

that is (p,q)-extension of the Eq. (5) in [18].

Conversely, if we consider the function $u(x) = x^3 + x^2$ or $u(x) = \cos x$, the quantity $u(px)$ and $u(qx)$ can not be derived in terms of u in a basic way, and thereby it is impossible to write a general chain rule.

As has been done for multiple q-calculus in [24], the case $D_{p,q}\varpi\left(x\right) = 0$ for any function is ϖ if and only if $\varpi\left(px\right) = \varpi\left(qx\right)$ called (p,q)-periodic function. By choosing $p = e^{\ln p}$ and $x = e^{\ln x} = e^y$ we can write $\varpi\left(x\right) = \varpi\left(e^y\right) \equiv G\left(y\right)$ and

$$\varpi\left(px\right) = G\left(y + \ln p\right).$$

Using condition of (p,q)-periodicity of $F\left(x\right)$, we get

$$G\left(y + \ln p\right) = G\left(y + \ln q\right),$$

and if we denote $y + \ln p = z$, then we realize

$$G(z) = G\left(z + \ln \frac{q}{p}\right),$$

which implies that $G(z)$ is a standart periodic function

$$G(z) = G(z + t),$$

with period $t = \ln \frac{q}{p} = \ln q - \ln p$.

Applying k times the derivative operator $D_{p,q}$ to $(a \ominus x)_{p,q}^n$ successively yields to the following proportion.

Proposition 2.1. *Let n be a positive integer and $0 \leq k \leq n$, we have*

$$D_{p,q}^k \frac{1}{(x \ominus a)_{p,q}^n} = (-1)^k q^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(q^k x \ominus a)_{p,q}^{n+k}}. \tag{9}$$

Similarly, we get the following result.

Proposition 2.2. *Let $n \geq 1$ be an integer and $0 \leq k \leq n$, we have*

$$D_{p,q}^k \frac{1}{(a \ominus x)_{p,q}^n} = p^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(a \ominus p^k x)_{p,q}^{n+k}}. \tag{10}$$

Now, we analyze some properties of the (p,q)-exponential functions. We observe that

$$e_{\frac{1}{p},\frac{1}{q}}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\frac{1}{p},\frac{1}{q}}!} p^{-\binom{n}{2}} x^n = \sum_{n=0}^{\infty} \frac{1}{[n]_{p,q}!} q^{\binom{n}{2}} x^n,$$

thus, we get

$$e_{\frac{1}{p},\frac{1}{q}}(x) = E_{p,q}(x). \tag{11}$$

In the same way, we have

$$E_{\frac{1}{p},\frac{1}{q}}(x) = e_{p,q}(x). \tag{12}$$

How about the additive property of the (p,q)-exponential functions? The answer is that the (p,q)-exponential functions do not have any additive propertties. Indeed,

$$
\begin{aligned}
e_{p,q}(x) E_{p,q}(y) &= \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{y^n}{[n]_{p,q}!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k y^{n-k}\right) \frac{1}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(x \oplus y)_{p,q}^n}{[n]_{p,q}!},
\end{aligned} \tag{13}
$$

which is not any form of (p,q)-exponential functions ($e_{p,q}(x)$ and $E_{p,q}(x)$).

We define a new type of (p,q)-exponential functions as

$$\widetilde{e}_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!}. \tag{14}$$

From Eqs. (13) and (14), we deduce

$$e_{p,q}(x) E_{p,q}(y) = \widetilde{e}_{p,q}(x \oplus y)_{p,q},$$

which can be called addition formula for (p,q)-exponentials. Similar to the usual exponential function $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ and the q-exponential function $e_q(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{[n]_q!}$, (p,q)-exponential function has the following derivative property

$$D_{p,q}\widetilde{e}_{p,q}(x) = \widetilde{e}_{p,q}(x).$$

Note that (p,q)-Pascal rules are given by (*cf.* [10])

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}, \tag{15}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}. \tag{16}$$

Here, we give a theorem about properties of the (p,q)-binomial coefficients.

Theorem 2.1. *Each (p,q)-binomial coefficient is a polynomial including the parameters p and q of degree $k(n-k)$ whose leading coefficient is 1.*

Proof. This theorem can be proved by the same method in the proof of Corollary 6.1. in [18]. □

Also, the coefficients in the polynomial expression of $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ are symmetric.

Note that the (p,q)-binomial coefficients also have combinatorial interpretations like q-binomial coefficients and usual binomial coefficients.

Definition 2.1. [25] *The function $F(x)$ is a q-antiderivative of $f(x)$ if $D_{p,q}F(x) = f(x)$. It is shown by*

$$\int f(x)\, d_{p,q}x.$$

The following proposition is a (p,q)-generalization of the Proposition 18.1 in the book [18].

Proposition 2.3. *Let $0 < q < p \le 1$. Then, any function $f(x)$ has at most one (p,q)-antiderivative which is continuous at $x = 0$, up to adding a constant.*

Proof. By using the similar proof's technicals in [18] for q-antiderivative, this theorem can be proved directly. □

Let us consider the following formula for the change of variable $u = u(x) = \lambda x^{\mu}$ with λ, μ being constants. If $F(x)$ is a (p,q)-antiderivative of $f(x)$, then we have

$$\int f(u)\, d_{p,q}u = F(u) = F(u(x)).$$

Using the expression (8) we have for any $\widehat{p}$ and $\widehat{q}$

$$\begin{aligned} F(u(x)) &= \int D_{\widehat{p},\widehat{q}}F(u(x))\, d_{\widehat{p},\widehat{q}}x \\ &= \int \left(D_{\widehat{p}^{\mu},\widehat{q}^{\mu}}F\right)(u(x)).D_{\widehat{p},\widehat{q}}u(x)\, d_{\widehat{p},\widehat{q}}x \\ &= \int \left(D_{\widehat{p}^{\mu},\widehat{q}^{\mu}}F\right)(u(x))\, d_{\widehat{p},\widehat{q}}u(x). \end{aligned}$$

If we take $\widehat{p} = p^{1/\mu}$ and $\widehat{q} = q^{1/\mu}$, then we have $D_{\widehat{p}^{\mu},\widehat{q}^{\mu}}F = D_{p,q}F = f$, and hence

$$\int f(u)\, d_{p,q}u = \int f(u(x))\, d_{p^{1/\mu},q^{1/\mu}}u(x).$$

This formula implies that $f(u(x)) D_{p^{1/\mu},q^{1/\mu}}u(x)$ is one of the (p,q)-antiderivatives of $f(x)$.

The (p, q)-integral is defined (see [25]) by

$$\int f(x) \, d_{p,q}x = (p - q) \sum_{k=0}^{\infty} \frac{q^k x}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} x\right).$$

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a formal power series. Applying (p, q)-integral to the both sides of $f(x)$ yields to

$$\int f(x) \, d_{p,q}x = \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{[k+1]_{p,q}} + C,$$

where C is a constant.

Some simple examples of (p, q)-integral are

$$\int (x \oplus y)_{p,q}^n \, d_{p,q}x = \frac{\left(\frac{x}{p} \oplus y\right)_{p,q}^{n+1}}{[n+1]_{p,q}} + C,$$

$$\int \widetilde{e}_{p,q}(x) \, d_{p,q}x = \widetilde{e}_{p,q}(x) + C,$$

$$\int e_{p,q}(x) \, d_{p,q}x = e_{p,q}\left(\frac{x}{p}\right) + C,$$

$$\int E_{p,q}(x) \, d_{p,q}x = E_{p,q}\left(\frac{x}{q}\right) + C.$$

3. (p, q)-Trigonometric functions

The (p, q)-analogues of the sine, cosine, tangent and cotangent functions can be defined in the same manner as their well known Euler expressions by means of the exponential functions.

Definition 3.1. *Let* $i = \sqrt{-1} \in \mathbb{C}$. *Then two pairs of* (p, q)-*trigonometric functions are defined by*

$\sin_{p,q} x := \dfrac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i},$	$Sin_{p,q} x := \dfrac{E_{p,q}(ix) - E_{p,q}(-ix)}{2i},$
$\cos_{p,q} x := \dfrac{e_{p,q}(ix) + e_{p,q}(-ix)}{2},$	$Cos_{p,q} x := \dfrac{E_{p,q}(ix) + E_{p,q}(-ix)}{2},$
$\tan_{p,q} x := \dfrac{\sin_{p,q} x}{\cos_{p,q} x},$	$Tan_{p,q} x := \dfrac{Sin_{p,q} x}{Cos_{p,q} x},$
$\cot_{p,q} x := \dfrac{\cos_{p,q} x}{\sin_{p,q} x},$	$Cot_{p,q} x := \dfrac{Cos_{p,q} x}{Sin_{p,q} x}.$

(17)

Using the identity (17), we have

$$\sin_{p,q} x \, Sin_{p,q} x = -\frac{e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(-ix) - 2}{4},$$

and

$$\cos_{p,q} x \, Cos_{p,q} x = \frac{e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(-ix) + 2}{4}.$$

Thus, we derive the following formula

$$\sin_{p,q} x \, Sin_{p,q} x + \cos_{p,q} x \, Cos_{p,q} x = 1,$$

which is the (p, q)-analogue of the well-known formula

$$\sin^2 x + \cos^2 x = 1.$$

By means of the (p,q)-trigonometric functions, the corresponding secant and cosecant functions are described as

$\sec_{p,q} x := \dfrac{1}{\cos_{p,q} x}$,	$\csc_{p,q} x := \dfrac{1}{\sin_{p,q} x}$,
$\text{Sec}_{p,q} x := \dfrac{1}{\text{Cos}_{p,q} x}$,	$\text{Csc}_{p,q} x := \dfrac{1}{\text{Sin}_{p,q} x}$.

Note that the (p,q)-tangent and (p,q)-cotangent functions coincide the following equalities

$$\tan_{p,q} x = \text{Tan}_{p,q} x \text{ and } \cot_{p,q} x = \text{Cot}_{p,q} x,$$

which are (p,q)-extension of the results in [9]. The two (p,q)-tangent functions are valid, however, one of these functions may be more useful than the other under special cases.

Now let us investigate (p,q)-derivatives of the (p,q)-trigonometric functions. By making use of (3.1), we readily see that

$$
\begin{aligned}
D_{p,q} \sin_{p,q} x &= D_{p,q}\left(\frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i}\right) \\
&= \frac{D_{p,q}e_{p,q}(ix) - D_{p,q}e_{p,q}(-ix)}{2i} \\
&= \frac{e_{p,q}(px) + e_{p,q}(-px)}{2} = \cos_{p,q}(px).
\end{aligned}
$$

By the same way, the others are stated as follows.

Theorem 3.1. *The (p,q)-derivative operator fulfils the following equalities*

$D_{p,q} \sin_{p,q} x = \cos_{p,q}(px),$	$D_{p,q}\text{Sin}_{p,q}x = \text{Cos}_{p,q}(qx),$
$D_{p,q} \cos_{p,q} x = -\sin_{p,q}(px),$	$D_{p,q}\text{Cos}_{p,q}x = -\text{Sin}_{p,q}(qx),$
$D_{p,q} \tan_{p,q} x = 1 + \tan_{p,q}(px)\tan_{p,q}(qx),$	$D_{p,q}\text{Tan}_{p,q}x = 1 + \text{Tan}_{p,q}(px)\,\text{Tan}_{p,q}(qx),$
$D_{p,q} \cot_{p,q} x = -\dfrac{\sin^2_{p,q}(px) + \cos^2_{p,q}(px)}{\sin_{p,q}(px)\sin_{p,q}(qx)},$	$D_{p,q}\text{Cot}_{p,q}x = -\dfrac{\text{Sin}^2_{p,q}(qx) + \text{Cos}^2_{p,q}(qx)}{\text{Sin}_{p,q}(px)\,\text{Sin}_{p,q}(qx)},$
$D_{p,q} \sec_{p,q} x = \sec_{p,q}(qx)\tan_{p,q}(px),$	$D_{p,q}\text{Sec}_{p,q}x = \text{Sec}_{p,q}(px)\,\text{Tan}_{p,q}(qx),$
$D_{p,q} \csc_{p,q} x = -\csc_{p,q}(qx)\cot_{p,q}(px),$	$D_{p,q}\text{Csc}_{p,q}x = -\text{Csc}_{p,q}(px)\,\text{Cot}_{p,q}(qx).$

Now (p,q)-integration properties of the (p,q)-cosine and (p,q)-sine functions are given as follows.

Theorem 3.2. *The following (p,q)-integrals are verified by easy computations:*

$\int \sin_{p,q} x\, d_{p,q}x = -\cos_{p,q}\left(\frac{x}{p}\right) + C,$
$\int \cos_{p,q} x\, d_{p,q}x = \sin_{p,q}\left(\frac{x}{p}\right) + C,$
$\int \text{Sin}_{p,q} x\, d_{p,q}x = -\text{Cos}_{p,q}\left(\frac{x}{q}\right) + C,$
$\int \text{Cos}_{p,q} x\, d_{p,q}x = \text{Sin}_{p,q}\left(\frac{x}{q}\right) + C.$

The (p,q)-exponential functions are related to the (p,q)-cosine and (p,q)-sine functions as

$$
\begin{aligned}
e_{p,q}(ix) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} i^n \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} p^{(2n-1)n} x^{2n} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} p^{(2n+1)n} x^{2n+1} \\
&= \cos_{p,q}(x) + i\sin_{p,q}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
E_{p,q}(ix) &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} i^n \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} q^{(2n-1)n} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{(2n+1)n} x^{2n+1} \\
&= Cos_{p,q}(x) + iSin_{p,q}(x).
\end{aligned}
$$

The following theorem includes a connection with (p,q)-sine and (p,q)-cosine functions.

Theorem 3.3. *We have*

$$
\sin_{p,q} x \ Cos_{p,q} x = \cos_{p,q} x \ Sin_{p,q} x.
$$

The (p,q)-analogues of hyperbolic functions can be defined in the same manner with their well known Euler expressions by means of the exponential functions.

Definition 3.2. *The (p,q)-hyperbolic functions are described as*

$$
\begin{array}{|c|c|}
\hline
\sinh_{p,q} x = \dfrac{e_{p,q}(x) - e_{p,q}(-x)}{2}, & Sinh_{p,q} x = \dfrac{E_{p,q}(x) - E_{p,q}(-x)}{2}, \\
\hline
\cosh_{p,q} x = \dfrac{e_{p,q}(x) + e_{p,q}(-x)}{2}, & Cosh_{p,q} x = \dfrac{E_{p,q}(x) + E_{p,q}(-x)}{2}, \\
\hline
\tanh_{p,q} x = \dfrac{\sinh_{p,q} x}{\cosh_{p,q} x}, & Tanh_{p,q} x = \dfrac{Sinh_{p,q} x}{Cosh_{p,q} x}, \\
\hline
\coth_{p,q} x = \dfrac{\cosh_{p,q} x}{\sinh_{p,q} x}, & Coth_{p,q} x = \dfrac{Cosh_{p,q} x}{Sinh_{p,q} x}. \\
\hline
\end{array}
\tag{18}
$$

The following relationhips hold true:

$$
\boxed{e_{p,q}(x) = \cosh_{p,q} x + \sinh_{p,q} x, \quad E_{p,q}(x) = Cosh_{p,q} x + Sinh_{p,q} x.}
$$

Using the identity (18), we have

$$
\sinh_{p,q} x \ Sinh_{p,q} x = \frac{e_{p,q}(x) E_{p,q}(x) + e_{p,q}(-x) E_{p,q}(-x) - 2}{4}
$$

and

$$
\cosh_{p,q} x \ Cosh_{p,q} x = \frac{e_{p,q}(x) E_{p,q}(x) + e_{p,q}(-x) E_{p,q}(-x) + 2}{4}.
$$

Hence, we observe that

$$
\cosh_{p,q} x \ Cosh_{p,q} x - \sinh_{p,q} x \ Sinh_{p,q} x = 1,
$$

which is the (p,q)-analogue of the well known formula

$$
\cosh^2 x - \sinh^2 x = 1.
$$

Following the (p,q)-hyperbolic functions, the hyperbolic (p,q)-secant and (p,q)-cosecant functions are defined by

$$
\begin{array}{|c|c|}
\hline
\operatorname{sech}_{p,q} x := \dfrac{1}{\cosh_{p,q} x}, & \operatorname{csch}_{p,q} x := \dfrac{1}{\sinh_{p,q} x}, \\
\hline
\operatorname{Sech}_{p,q} x := \dfrac{1}{Cosh_{p,q} x}, & \operatorname{Csch}_{p,q} x := \dfrac{1}{Sinh_{p,q} x}. \\
\hline
\end{array}
$$

The following theorem consists of the (p,q)-derivative properties of (p,q)-hyperbolic functions.

Theorem 3.4. *The (p, q)-derivative operator fulfils the following equations*

$D_{p,q} \sinh_{p,q} x = \cosh_{p,q}(px),$	$D_{p,q} Sinh_{p,q}x = Cosh_{p,q}(qx),$
$D_{p,q} \cosh_{p,q} x = \sinh_{p,q}(px),$	$D_{p,q} Cosh_{p,q}x = Sinh_{p,q}(qx),$
$D_{p,q} \tanh_{p,q} x = 1 - \tanh_{p,q}(px)\tanh_{p,q}(qx),$	$D_{p,q} Tanh_{p,q}x = 1 - Tanh_{p,q}(px)\,Tanh_{p,q}(qx),$
$D_{p,q} \coth_{p,q} x = \dfrac{\sinh_{p,q}^2(px) - \cosh_{p,q}^2(px)}{\sinh_{p,q}(px)\sinh_{p,q}(qx)},$	$D_{p,q} Coth_{p,q}x = \dfrac{Sinh_{p,q}^2(qx) - Cosh_{p,q}^2(qx)}{Sinh_{p,q}(px)\,Sinh_{p,q}(qx)},$
$D_{p,q} sech_{p,q}x = -sech_{p,q}(qx)\tanh_{p,q}(px),$	$D_{p,q} Sech_{p,q}x = -Sech_{p,q}(px)\,Tanh_{p,q}(qx),$
$D_{p,q} csch_{p,q}x = -csch_{p,q}(qx)\coth_{p,q}(px),$	$D_{p,q} Csch_{p,q}x = -Csch_{p,q}(px)\,Coth_{p,q}(qx).$

The following theorem includes the (p, q)-integral properties of (p, q)-hyperbolic functions.
Theorem 3.5. *We have*

$\int \sinh_{p,q}(x)\, d_{p,q}x = \cosh_{p,q}\left(\frac{x}{p}\right) + C,$
$\int \cosh_{p,q}(x)\, d_{p,q}x = \sinh_{p,q}\left(\frac{x}{p}\right) + C,$
$\int Sinh_{p,q}(x)\, d_{p,q}x = Cosh_{p,q}\left(\frac{x}{q}\right) + C,$
$\int Cosh_{p,q}(x)\, d_{p,q}x = Sinh_{p,q}\left(\frac{x}{q}\right) + C.$

The (p, q)-exponential functions are related to the (p, q)- hyperbolic cosine and (p, q)- hyperolic sine functions by

$$
\begin{aligned}
e_{p,q}(x) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} p^{(2n-1)n} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} p^{(2n+1)n} x^{2n+1} \\
&= \cosh_{p,q}(x) + \sinh_{p,q}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
E_{p,q}(x) &= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} q^{(2n-1)n} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{(2n+1)n} x^{2n+1} \\
&= Cosh_{p,q}x + Sinh_{p,q}x.
\end{aligned}
$$

Using the formulas (11) and (12), we obtain the following results.

Theorem 3.6. *The following identities are readily attested:*

$\sin_{\frac{1}{p},\frac{1}{q}} x = Sin_{p,q}x,$	$Sin_{\frac{1}{p},\frac{1}{q}} x = \sin_{p,q} x,$
$\cos_{\frac{1}{p},\frac{1}{q}} x = Cos_{p,q}x,$	$Cos_{\frac{1}{p},\frac{1}{q}} x = \cos_{p,q} x,$
$\tan_{\frac{1}{p},\frac{1}{q}} x = Tan_{p,q}x,$	$Tan_{\frac{1}{p},\frac{1}{q}} x = \tan_{p,q} x,$
$\cot_{\frac{1}{p},\frac{1}{q}} x = Cot_{p,q}x,$	$Cot_{\frac{1}{p},\frac{1}{q}} x = \cot_{p,q} x,$
$\sec_{\frac{1}{p},\frac{1}{q}} x = Sec_{p,q}x,$	$Sec_{\frac{1}{p},\frac{1}{q}} x = \sec_{p,q} x,$
$\csc_{\frac{1}{p},\frac{1}{q}} x = Csc_{p,q}x,$	$Csc_{\frac{1}{p},\frac{1}{q}} x = \csc_{p,q} x,$
$\sinh_{\frac{1}{p},\frac{1}{q}} x = Sinh_{p,q}x,$	$Sinh_{\frac{1}{p},\frac{1}{q}} x = \sinh_{p,q} x,$
$\cosh_{\frac{1}{p},\frac{1}{q}} x = Cosh_{p,q}x,$	$Cosh_{\frac{1}{p},\frac{1}{q}} x = \cosh_{p,q} x,$
$\tanh_{\frac{1}{p},\frac{1}{q}} x = Tanh_{p,q}x,$	$Tanh_{\frac{1}{p},\frac{1}{q}} x = \tanh_{p,q} x,$
$\coth_{\frac{1}{p},\frac{1}{q}} x = Coth_{p,q}x,$	$Coth_{\frac{1}{p},\frac{1}{q}} x = \coth_{p,q} x,$
$sech_{\frac{1}{p},\frac{1}{q}}x = Sech_{p,q}x,$	$Sech_{\frac{1}{p},\frac{1}{q}}x = sech_{p,q}x,$
$csch_{\frac{1}{p},\frac{1}{q}}x = Csch_{p,q}x,$	$Csch_{\frac{1}{p},\frac{1}{q}}x = csch_{p,q}x.$

In terms of the new (p,q)-exponential function $\widetilde{e}_{p,q}x$, the corresponding trigonometric functions can be defined as

$\widetilde{\sin}_{p,q}x := \dfrac{\widetilde{e}_{p,q}(ix) - \widetilde{e}_{p,q}(-ix)}{2i},$	$\widetilde{\tan}_{p,q}x := \dfrac{\widetilde{\sin}_{p,q}x}{\widetilde{\cos}_{p,q}x},$
$\widetilde{\cos}_{p,q}x := \dfrac{\widetilde{e}_{p,q}(ix) + \widetilde{e}_{p,q}(-ix)}{2},$	$\widetilde{\cot}_{p,q}x := \dfrac{\widetilde{\cos}_{p,q}x}{\widetilde{\sin}_{p,q}x},$
$\widetilde{\sec}_{p,q}x = \dfrac{1}{\widetilde{\cos}_{p,q}x},$	$\widetilde{\csc}_{p,q}x = \dfrac{1}{\widetilde{\sin}_{p,q}x},$

which satisfy

$D_{p,q}\widetilde{\sin}_{p,q}x = \widetilde{\cos}_{p,q}x,$
$D_{p,q}\widetilde{\cos}_{p,q}x = -\widetilde{\sin}_{p,q}x,$
$D_{p,q}\widetilde{\tan}_{p,q}x = \dfrac{\widetilde{\cos}_{p,q}x + \widetilde{\tan}_{p,q}(px)\,\widetilde{\sin}_{p,q}x}{\widetilde{\cos}_{p,q}(qx)},$
$D_{p,q}\widetilde{\cot}_{p,q}x = -\dfrac{\widetilde{\sin}_{p,q}x + \widetilde{\cot}_{p,q}(px)\,\widetilde{\cos}_{p,q}x}{\widetilde{\sin}_{p,q}(qx)},$
$D_{p,q}\widetilde{\sec}_{p,q}x = \dfrac{\widetilde{\sin}_{p,q}x}{\widetilde{\cos}_{p,q}(px)\,\widetilde{\cos}_{p,q}(qx)},$
$D_{p,q}\widetilde{\csc}_{p,q}x = -\dfrac{\widetilde{\cos}_{p,q}x}{\widetilde{\sin}_{p,q}(px)\,\widetilde{\sin}_{p,q}(qx)}.$

The (p,q)-exponential functions $\widetilde{e}_{p,q}x$ are correlated with the new (p,q)-cosine and (p,q)-sine functions as

$$
\begin{aligned}
\widetilde{e}_{p,q}(ix) &= \sum_{n=0}^{\infty} i^n \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} x^{2n+1} \\
&= \widetilde{\cos}_{p,q}(x) + i\widetilde{\sin}_{p,q}(x).
\end{aligned}
$$

In terms of the new (p,q)-exponential functions $\widetilde{e}_{p,q}x$, the corresponding hyperbolic functions are described as

$\widetilde{\sinh}_{p,q}x := \dfrac{\widetilde{e}_{p,q}(x) - \widetilde{e}_{p,q}(-x)}{2},$	$\widetilde{\tanh}_{p,q}x := \dfrac{\widetilde{\sinh}_{p,q}x}{\widetilde{\cosh}_{p,q}x},$
$\widetilde{\cosh}_{p,q}x := \dfrac{\widetilde{e}_{p,q}(x) + \widetilde{e}_{p,q}(-x)}{2},$	$\widetilde{\coth}_{p,q}x := \dfrac{\widetilde{\cosh}_{p,q}x}{\widetilde{\sinh}_{p,q}x},$
$\widetilde{\operatorname{sech}}_{p,q}x = \dfrac{1}{\widetilde{\cosh}_{p,q}x},$	$\widetilde{\operatorname{csch}}_{p,q}x = \dfrac{1}{\widetilde{\sinh}_{p,q}x},$

which fulfil

$D_{p,q}\widetilde{\sinh}_{p,q}x = \widetilde{\cosh}_{p,q}x,$
$D_{p,q}\widetilde{\cosh}_{p,q}x = \widetilde{\sinh}_{p,q}x,$
$D_{p,q}\widetilde{\tanh}_{p,q}x = \dfrac{\widetilde{\cosh}_{p,q}x + \widetilde{\tanh}_{p,q}(px)\,\widetilde{\sinh}_{p,q}x}{\widetilde{\cosh}_{p,q}(qx)},$
$D_{p,q}\widetilde{\coth}_{p,q}x = -\dfrac{\widetilde{\sinh}_{p,q}x - \widetilde{\coth}_{p,q}(px)\,\widetilde{\cosh}_{p,q}x}{\widetilde{\sinh}_{p,q}(qx)},$
$D_{p,q}\widetilde{\operatorname{sech}}_{p,q}x = -\dfrac{\widetilde{\sinh}_{p,q}x}{\widetilde{\cosh}_{p,q}(px)\,\widetilde{\cosh}_{p,q}(qx)},$
$D_{p,q}\widetilde{\operatorname{csch}}_{p,q}x = -\dfrac{\widetilde{\cosh}_{p,q}x}{\widetilde{\sinh}_{p,q}(px)\,\widetilde{\sinh}_{p,q}(qx)}.$

Theorem 3.7. *The following (p,q)-integrals are valid:*

$\int \widetilde{\sin}_{p,q}x\,d_{p,q}x = -\widetilde{\cos}_{p,q}x + C,$
$\int \widetilde{\cos}_{p,q}x\,d_{p,q}x = \widetilde{\sin}_{p,q}x + C,$
$\int \widetilde{\sinh}_{p,q}x\,d_{p,q}x = \widetilde{\cosh}_{p,q}x + C,$
$\int \widetilde{\cosh}_{p,q}x\,d_{p,q}x = \widetilde{\sinh}_{p,q}x + C.$

New (p,q)-trigonometric functions can be expressed by earlier defined (p,q)-trigonometric functions as follows.

Theorem 3.8. *The following equalities*

$\widetilde{\sin}_{p,q}(x \oplus x)_{p,q} = \sin_{p,q}x\;Cos_{p,q}x + \cos_{p,q}x Sin_{p,q}x,$
$\widetilde{\sin}_{p,q}(x \ominus x)_{p,q} = \sin_{p,q}x\;Cos_{p,q}x - \cos_{p,q}x Sin_{p,q}x,$
$\widetilde{\cos}_{p,q}(x \oplus x)_{p,q} = \cos_{p,q}x\;Cos_{p,q}x - \sin_{p,q}x Sin_{p,q}x,$
$\widetilde{\cos}_{p,q}(x \ominus x)_{p,q} = \cos_{p,q}x\;Cos_{p,q}x + \sin_{p,q}x Sin_{p,q}x,$
$\sinh_{p,q}(x \oplus x)_{p,q} = \sinh_{p,q}x\,Cosh_{p,q}x + \cosh_{p,q}x Sinh_{p,q}x,$
$\sinh_{p,q}(x \ominus x)_{p,q} = \sinh_{p,q}x\,Cosh_{p,q}x - \cosh_{p,q}x Sinh_{p,q}x,$
$\cosh_{p,q}(x \oplus x)_{p,q} = \cosh_{p,q}x\,Cosh_{p,q}x + \sinh_{p,q}x Sinh_{p,q}x,$
$\cosh_{p,q}(x \ominus x)_{p,q} = \cosh_{p,q}x\,Cosh_{p,q}x - \sinh_{p,q}x Sinh_{p,q}x$

are true.

The following intriguing identities between (p,q)-trigonometric and (p,q)-hyperbolic functions hold true.

Theorem 3.9. *We have*

$\sinh_{p,q} x = -i \sin_{p,q}(ix)$ $Sinh_{p,q}x = -i Sin_{p,q}(ix)$ $\widetilde{\sinh}_{p,q}x = -i\widetilde{\sin}_{p,q}(ix)$	$q \to p = 1$	$\sinh x = -i \sin(ix),$
$\cosh_{p,q} x = \cos_{p,q}(ix)$ $Cosh_{p,q}x = Cos_{p,q}(ix)$ $\widetilde{\cosh}_{p,q}x = \widetilde{\cos}_{p,q}(ix)$	$q \to p = 1$	$\cosh x = \cos(ix),$
$\tanh_{p,q} x = -i \tan_{p,q}(ix)$ $Tanh_{p,q}x = -i Tan_{p,q}(ix)$ $\widetilde{\tanh}_{p,q}x = -i\widetilde{\tan}_{p,q}(ix)$	$q \to p = 1$	$\tanh x = -i \tan(ix),$
$\coth_{p,q} x = i \cot_{p,q}(ix)$ $Coth_{p,q}x = i Cot_{p,q}(ix)$ $\widetilde{\coth}_{p,q}x = i\widetilde{\cot}_{p,q}(ix)$	$q \to p = 1$	$\coth x = i \cot(ix).$

Here the (p,q)-trigonometric functions and the (p,q)-hypergeometric functions are examined whether these functions are odd functions or even functions.

Theorem 3.10. *We have*

$\sin_{p,q}(-x) = -\sin_{p,q}x,$	$Sin_{p,q}(-x) = -Sin_{p,q}x,$	$\widetilde{\sin}_{p,q}(-x) = -\widetilde{\sin}_{p,q}x,$
$\cos_{p,q}(-x) = \cos_{p,q}x,$	$Cos_{p,q}(-x) = Cos_{p,q}x,$	$\widetilde{\cos}_{p,q}(-x) = \widetilde{\cos}_{p,q}x,$
$\tan_{p,q}(-x) = -\tan_{p,q}x,$	$Tan_{p,q}(-x) = -Tan_{p,q}x,$	$\widetilde{\tan}_{p,q}(-x) = -\widetilde{\tan}_{p,q}x,$
$\cot_{p,q}(-x) = -\cot_{p,q}x,$	$Cot_{p,q}(-x) = -Cot_{p,q}x,$	$\widetilde{\cot}_{p,q}(-x) = -\widetilde{\cot}_{p,q}x,$
$\sec_{p,q}(-x) = \sec_{p,q}x,$	$Sec_{p,q}(-x) = Sec_{p,q}x,$	$\widetilde{\sec}_{p,q}(-x) = \widetilde{\sec}_{p,q}x,$
$\csc_{p,q}(-x) = -\csc_{p,q}x,$	$Csc_{p,q}(-x) = -Csc_{p,q}x,$	$\widetilde{\csc}_{p,q}(-x) = -\widetilde{\csc}_{p,q}x,$
$\sinh_{p,q}(-x) = -\sinh_{p,q}x,$	$Sinh_{p,q}(-x) = -Sinh_{p,q}x,$	$\widetilde{\sinh}_{p,q}(-x) = -\widetilde{\sinh}_{p,q}x,$
$\cosh_{p,q}(-x) = \cosh_{p,q}x,$	$Cosh_{p,q}(-x) = Cosh_{p,q}x,$	$\widetilde{\cosh}_{p,q}(-x) = \widetilde{\cosh}_{p,q}x,$
$\tanh_{p,q}(-x) = -\tanh_{p,q}x,$	$Tanh_{p,q}(-x) = -Tanh_{p,q}x,$	$\widetilde{\tanh}_{p,q}(-x) = -\widetilde{\tanh}_{p,q}x,$
$\coth_{p,q}(-x) = -\coth_{p,q}x,$	$Coth_{p,q}(-x) = -Coth_{p,q}x,$	$\widetilde{\coth}_{p,q}(-x) = -\widetilde{\coth}_{p,q}x,$
$sech_{p,q}(-x) = sech_{p,q}x,$	$Sech_{p,q}(-x) = Sech_{p,q}x,$	$\widetilde{sech}_{p,q}(-x) = \widetilde{sech}_{p,q}x,$
$csch_{p,q}(-x) = -csch_{p,q}x,$	$Csch_{p,q}(-x) = -Csch_{p,q}x,$	$\widetilde{csch}_{p,q}(-x) = -\widetilde{csch}_{p,q}x.$

4. Conclusion

We have obtained some new results for *post-quantum calculus*, denoted by (p,q)-calculus. Our study in content is as follows: a chain rule for (p,q)-derivative has been given, a new (p,q)-analogue of the exponential functions has been introduced and some of its properties including the addition property for (p,q)-exponential functions have been investigated. Several useful results involving (p,q)binomial coefficients and (p,q)-anti-derivatives been discovered. Finally, (p,q)-analogue of some elementary functions including trigonometric functions and hyperbolic functions have been listed and some properties and relations among them have been mentioned.

REFERENCES

[1] Acar, T., Aral, A., Mohiuddine, S.A., (2016), On Kantorovich modification of (p,q)-Baskakov operators, J. Inequal. Appl., 98, 14 pp.

[2] Araci, S., Duran, U., Acikgoz, M., (2017), (p,q)-Volkenborn integral, J. Number Theory, 171, pp.18-30.

[3] Araci, S., Duran, U., Acikgoz, M., Srivastava, H.M., (2016), A certain (p,q)-derivative operator and associated divided differences, Journal of Inequalities and Applications, 301, 8 pp.

[4] Arik, M., Demircan, E., Turgut, T., Ekinci, L., Mungan, M., (1992), Fibonacci oscillators, Z. Phys. C-Particles and Fields, 55, pp.89-95.

[5] Brodimas, G., Jannussis, A., Mignani, R., (1991), Two-parameter Quantum Groups, Universita di Roma Preprint Nr., 820.

[6] Burban, I.M., Klimyk, A.U., (1994), P,Q-differentiation, P,Q-integration and P,Q-hypergeometric functions related to quantum groups, Integral Transforms Spec. Funct., 2(1), pp.15-36.

[7] Chakrabarti, R., Jagannathan, R., (1991), A (p,q)-oscillator realization of two-parameter quantum algebras, J. Phys. A: Math. Gen., 24(7), L711-L718.

[8] Cheon, G.S., (2003), A note on the Bernoulli and Euler polynomials, Appl. Math. Lett., 16(3), pp.365-368.

[9] Cieśliński, J.L., (2010), Improved q-exponential and q-trigonometric functions, arXiv:1006.5652v1 [math.CA].

[10] Corcino, R.B., (2008), On P,Q-Binomial coefficients , Electron. J. Combin. Number Theory, 8(2008).

[11] Duran, U., Acikgoz, M., Araci, S., (2016), On some polynomials derived from (p,q)-calculus, J. Comput. Theor. Nanosci., 13(11), pp.7903-7908.

[12] Duran, U., Acikgoz, M., Araci, S., (2018), Research on some new results arising from multiple q-calculus, Filomat, 32(1), pp.1-9.

[13] Duran, U., Acikgoz, M., Araci, S., (2016), On (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials, J. Comput. Theor. Nanosci., 13(11), pp.7833-7846.

[14] Ernst, T., (2000), The history of q-calculus and a new method, U.U.D.M. Department of Mathematics, Uppsala University, Uppsala Report 2000.

[15] Jackson, F.H., (1910), On q-definite integrals, Quart J. Pure Appl. Math., 41, pp.193-203.

[16] Jagannathan, R., Rao, K.S., (2005), Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, in Proceeding of the International Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre, Kumbakonam, India, 20-21 December.

[17] Jagannathan, R., (1997), (P,Q)-Special functions, Special Functions and Differential Equations, Proceedings of a Workshop held at The Institute of Mathematical Sciences, Matras, India, pp.13-24.

[18] Kac, V., Cheung, P., (2002), Quantum Calculus, New York: Springer.

[19] Kim, T., (2007), q-Extension of the Euler formula and trigonometric functions, Russ. J. Math. Phys., 14(3), pp.275-278.

[20] Mursaleen, M., Ansari, K.J., Khan, A., (2015), On (p,q)-analogue of Bernstein operators, Appl. Math. Comput., 266, pp.874-882.

[21] Mursaleen, M., Alotaibi, A., Ansari, KH., (2016), On a Kantorovich variant of (p,q)-Szász-Mirakjan operators, J. Funct. Spaces, Article ID 1035253, 9 pp.

[22] Mursaleen, M., Ansari, KJ., Khan, A., (2015), Some approximation results by (p,q)-analogue of Bernstein-Stancu operators, Appl. Math. Comput., 264, pp.392-402.

[23] Mursaleen, M., Nasiuzzaman, M., Nurgali, A., (2015), Some approximation results on Bernstein-Schurer operators defined by (p,q)-integers, J. Inequal. Appl., Article ID 249, 12 pp.

[24] Nalci, S., Nalci, O.K., (2014), Exactly Solvable q-Extended Nonlinear Classical and Quantum Models, Lambert Academic Publishing, 272 pp.

[25] Sadjang, P.N., (2013), On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas, arXiv:1309.3934 [math.QA].

[26] Sadjang, P.N., (2015), On the (p,q)-Gamma and the (p,q)-Beta functions, arXiv 1506.07394v1 [math.QA].

[27] Srivastava, H.M., Pinter, A., (2004), Remarks on some relationships between the bernoulli and euler polynomails, Appl. Math. Lett., 17, pp.375-380.

[28] Wachs, M., White, D., (1991), (p,q)-Stirling numbers and set partition statistics, J. Combin. Theory Ser, A. 56(1), pp.27-46.

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